

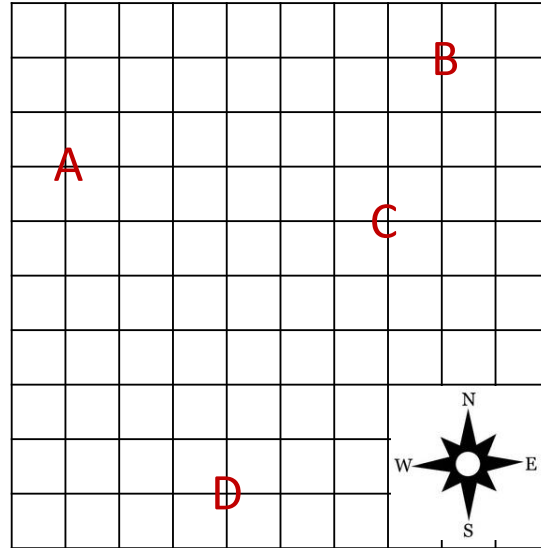
A Placement Problem

(no, we will not be using test scores)

Four towns are building a park together.

- Town A is 3 miles North and 1.5 miles West of town D.
- Town B is 4 miles North and 2 miles East of town D.
- Town C is 2.5 miles North and 1.5 miles East of town D.

Where should they build the park?



What else would you like to know in order to make a better decision?

Notes

General arc of the session

Discussion of the teaser problem will be used as a launching pad for modeling: we are starting with an easy problem that introduces both the mathematical ideas we will use (distance, weighted average, proportional relationships) and the bigger context of a real-world situation. First, we will discuss the teaser problem from the flyer to introduce the concept of “center of mass” or “average of the coordinates” or “average distance”, etc.

Then, I will ask what other information participants would want to take into account if this was a real-life situation. Depending on what participants come up with, this might lead us into generalizing the center of mass idea to a weighted average. We can also (or instead) introduce the concept of a “happiness function”, or how to put together various interests to get comparable numbers on which park might be better.

By doing this, I will “model” what we do in mathematical modeling: we make assumptions as to what matters, we use math to quantify how and how much this matters, and we increase the complexity of the model if we decide to take into account additional, relevant information. But I won’t tell them that until the end.

We will then be ready to work on the real-world example!

Towards the end, I will keep some time to formalize the steps of modeling, and make an explicit link to the practice standard of modeling in the Common Core standards. I will also give resources, and tell participants about high school modeling contests!

The moral of the story: modeling a real world situation is possible, and not so daunting if we can do this step by step, abstracting away some information and choosing to focus on other, manageable aspects. The more mathematical tools we can use, the more complex, precise and useful our model might be.

Materials to bring:

- Computer, adapter, charger, pdf of slides.
- Printed notes.
- Map of cities with selected parks marked (a few copies).
- Report on selected parks (a few copies).
- **Protractor, compass** (see notes below).

Notes post-session:

- **The teaser problem itself could have taken the whole session! Participants were quite interested:**
 - **One team looked at a one-dimensional version with one person at location 0, 2 people at location 1, and the park at location x ($0 \leq x \leq 1$). Later on, instead of working on the real-world problem, they stuck to this one-dimensional example, and used Geogebra to explore the minimum for x in $[0, 1]$ of functions $|x|^\alpha + 2|1 - x|^\alpha$, for various α 's.**

- Quite a few participants thought of angle or median bisectors (as one often does for triangles), but for the quadrilateral of the cities. For those participants, having compasses and protractors would have been helpful (they were only given rulers and quad paper). They even thought of approximating the quadrilateral by a triangle because of its special shape.
- Some participants even explored taking the four possible subsets of 3 towns out of the 4. For each such subset of 3 towns, they drew a triangle, and found the centroid of that triangle. The four centroids then make another quadrilateral, smaller than the first. As you keep going, you might converge to a single point, and this is where they wanted to put the park (they didn't actually go through with this, but found a website with an animation of what they wanted to do).
- Participants had no trouble at all coming up with other information they would need to take into account.
- I didn't talk about weighted averages using population as in the notes on next page, but went straight to the real problem. (We had already spent more than a half hour on the teaser.) This seems to have left some participants feeling overwhelmed with the task, and it was very hard for me after that to encourage them to just pick one thing they thought would be relevant and only take that into account. Maybe a different set-up (“choose a single relevant thing other than location and make a decision on where to put the park”) would have been more beneficial to some participants. **Another way to do this could be to ask them to take a minute and think individually of the most important thing they would like to take into account. Then participants could make teams with others who had a similar priority.**
- Most participants were reluctant to actually come up with a “happiness function” or “points” to give different locations, and only used words to justify their decision, even when prodded. Again, working first on the weighted averages with population might have helped.
- Some resistance also came from participants who said that, knowing the political context of these towns, there is no way they would get together to fund a park. There also seemed to be quite a bit of park space already, and the new proposed parks were quite small. Some said they were so small they would not attract people from very far, so again, there was no point for the towns to all pay for this. They then thought of building say a skating rink or something more special (a new high school, a youth or community center), that might actually draw people from all four towns.
- I didn't go over the modeling framework (slide 9), partly from lack of time and partly because I felt that at least the participants who hadn't done much modeling today might not be interested. But maybe they would have benefited from it?

More detailed plan

- Teaser problem:
 - Ask what they came up with, how they found their solution, etc.
 - We might have to break it down a little and talk about only two towns at first, where you could put the park in that case. This might lead more naturally to the concept of an average, for each coordinate (x and y). (I could bring a large piece of cardboard where I have drawn a dot for each city, fix equal weights on the cardboard for each city, and find the center of mass with my finger.)
 - If we take town D as the origin, there might be confusion as to dividing by 3 or 4 for the average (town D is at a distance of 0). Go back to the two-city problem if needed.
 - Alternately: average of towns A and C, then B and D. Then average those two averages!
 - Now, discuss population. Should it change the answer? If so, how? Hopefully this will lead to the concept of a weighted average. (Back to the large piece of cardboard: now change the weights, and find the center of mass again. Or, imagine each person in town A is a distance a away from town D, so there are 20,000 people who are at a distance of a , 45,000 at a distance b , etc...)
 - If it comes up, discuss whether absolute vs relative weights matter. This might come up later with the real-world problem.
 - We have added more information to the problem, more complexity. In real life, things are even more complex! What else would you need to know?
- Discussion on “more information”:
 - Depending on how participation goes (hard to think of things? not sure whether something might be important or not?), I might ask them to do this in “Think-Pair-Share” so they can share with neighbor before sharing with the group. Can also phrase this as: what else could be relevant? What else should we take into account?
 - Possible answers: roads (speed of travel, red lights, shortest path using an online map program like Google Maps); the park cannot realistically be put anywhere (location might change size and amenities of park depending on budget); accessible by public transportation, number of parking spots if any; nice location (by a river, by an industrial park), population density (not everyone lives exactly in the center of town); others?

- Real-world problem:
 - Introduce the real-world problem: the cities of Waltham, Belmont, Watertown and Newton have decided to build a park together. They have found only three locations (see last page, which might be used to communicate that info to participants) as viable choices for where to put the park. They will be meeting later on this afternoon to make they final choice. Can you help them decide on which place would make their constituents the happiest?
 - Print a map for them with town and park locations, and a scale.
 - Focus changes from a weighted average for the location to a weighted average for the happiness of people. Do we want total happiness higher, or similar happiness for each town? Can let participants come up with these questions, or tackle whichever they prefer. Or help them think about it as I go around tables.
 - Have participants think about comparing their weighted averages to another team's, and how weights that total 1 (or 100...) allows easier (more fair) comparison.
 - If needed, encourage participants to add layers of complexity one at a time (like we did in the teaser). Start with the simplest, and gradually build it up. Useful especially if participants are daunted by the task and do not know where to start. Ask them “What is the one thing you think matters the most for happiness?”, and have them start with that.
 - Back-up problems, simpler problems, more advanced problems, side explorations:
 - * Belmont and Newton are outliers, so Watertown and Waltham decide to build their own park without the two other towns. Where should they put it?
 - * How do programs like Google Maps or a GPS manage to find the path with shortest travel time? What information do they need to calculate such things? Can you come up with a possible model for finding travel times?
 - * If you were allowed to build the park anywhere, where would you put it (so back to teaser problem, but need to take roads into account).
 - * Think of non-linear happiness functions (happiness grows with the square of surface area, or is inversely proportional to the distance from the park, etc).
 - * Not everyone lives in the center of each town.
 - * Are parks for children? Does that change our population assumption?
- What is modeling? How do we model a situation?
 - Towards the end, we can be explicit about what we did today:
 - * Make assumptions and approximations.
 - * Identify important quantities and their relationships.
 - * Apply the math we know to analyse those relationships.
 - * Interpret the mathematical results, reflect on whether they make sense.
 - * Improve the model by adjusting it or increasing its complexity.
 - Bring it back to the modeling practice standard of the Common Core, check off everything from the standard we actually did today!

Report on park locations — total budget of \$1.5 million

All costs are approximate.

1. MBTA lot in front of 50 Water street, Watertown.
 - Surface area: 2800 m².
 - Cost of buying land: \$300,000 (half of amount needed for MBTA to modernize another lot nearby and put their buses there instead).
 - Cost of landscaping: \$500,000.
 - Cost of cleaning: \$500,000.
 - Left over for infrastructure: \$200,000.

2. 427 River street, Waltham (old Destefano Bakery & Deli and adjacent vacant lot).
 - Surface area: 2400 m².
 - Cost of buying land: \$250,000 (legal fees for applying eminent domain law).
 - Cost of landscaping: \$250,000.
 - Cost of cleaning: unknown.
 - Left over for infrastructure: \$1,000,000?

3. Starting at 150 Woodland street in Belmont, North-East side of street up until train tracks. Possibility of adding the lot between the tracks and the street connecting Woodland to Prince street.
 - Surface area: 4000 m² (plus extra 2000 m² possible).
 - Cost of buying land: \$500,000 (plus \$250,000 possible).
 - Cost of landscaping: \$300,000 (plus \$150,000 possible).
 - Cost of cleaning: \$100,000 to 200,000 (plus \$50,000 to \$100,000 possible).
 - Left over for infrastructure: \$500,000 to 600,000 (or \$0 to \$150,000 left).