Partitioned Low Rank Compression of Absorbing Boundary Conditions for the Helmholtz equation

Rosalie Bélanger-Rioux, Harvard University

Laurent Demanet, Massachusetts Institute of Technology
Imaging and Computing Group

CSE 15
March 15, 2015

Helmholtz equation in unbounded domain

2D Helmholtz equation

$$\Delta u(\mathbf{x}) \; + \; \frac{\omega^2}{c^2(\mathbf{x})} \, u(\mathbf{x}) \; = \; f(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Solution u, frequency ω, medium $c(\mathbf{x})$, source $f(\mathbf{x})$.

Many sources!

Select outgoing waves using the Sommerfeld Radiation Condition

$$\lim_{r \to \infty} r^{1/2} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \qquad k = \frac{\omega}{c},$$

where r is the radial coordinate.

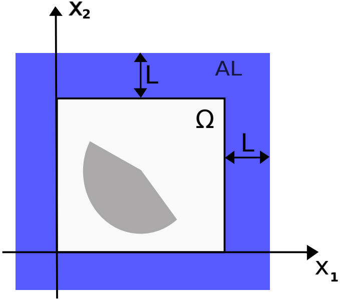
Applications

- Wave-based imaging, an inverse problem.
  - Seismic imaging: for rock formations.
  - Ultrasonic testing: non-destructive testing of objects for defects.
  - Ultrasonic imaging: visualizing a fetus, muscle, tendon or organ.
  - Synthetic-aperture radar imaging: visualizing a scene or detecting the presence of an object far away or through clouds, foliage.
- Photonics: studying the optical properties of crystals.
- Speeding up Domain Decomposition Methods.

Absorbing Boundary Conditions (ABCs) and Layers (ALs)

$$\Delta u(\mathbf{x}) + \frac{\omega^2}{c^2(\mathbf{x})} u(\mathbf{x}) = f(\mathbf{x}), \qquad k = \frac{\omega}{c(\mathbf{x})}, \qquad \mathbf{x} \in \Omega.$$

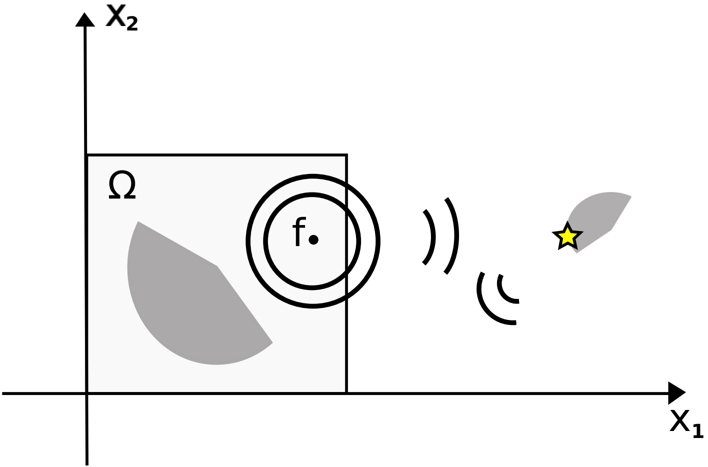
- Close system using Absorbing Boundary Condition (ABC) or Absorbing Layer (AL).
- N pts per dimension, $h = 1/N$.



Issue: absorbing layers tend to get thick in heterogeneous media.

Absorbing Layers in heterogeneous media

Physical width $L > 1$ or width in number of points $w > N$.



Our numerical scheme

Goal: Compress costly ABC or AL to speed up Helmholtz solver

Step 1: Obtain the exterior Dirichlet-to-Neumann (DtN) map D

- Matrix probing with solves of exterior problem

Step 2: Obtain a fast algorithm for matrix-vector products of D

- Partitioned low-rank (PLR) matrices, compress off-diagonal blocks

$$D \xrightarrow[\text{expansion}]{\text{probing}} \tilde{D} \xrightarrow[\text{compression}]{\text{PLR}} \overline{D}$$

Step 1: Obtain the exterior DtN map D

$$D \xrightarrow[\text{expansion}]{\text{probing}} \tilde{D} \xrightarrow[\text{compression}]{\text{PLR}} \overline{D}$$

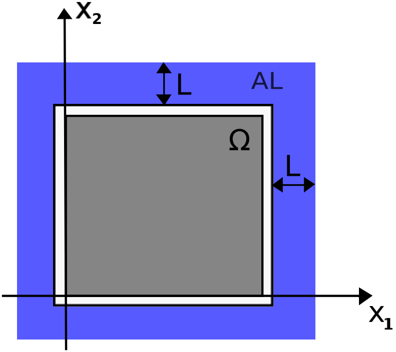
The exterior problem to obtain the exterior DtN map

$$\Delta u(\mathbf{x}) + \frac{\omega^2}{c^2(\mathbf{x})} u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2 \setminus \Omega$$

- $u(\mathbf{x}) = g(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$.
- Use ABC or AL.
- Solution u_1 on 1^{st} layer outside Ω.
- Obtain product of D with g:

$$Dg = \frac{u_1 - g}{h}.$$

- Use D in a Helmholtz solver instead of ABC or AL.



Matrix probing

$$M \in \mathbb{C}^{N \times N}, \qquad \text{single random vector } z$$

- Given: z and Mz
- Problem: recover M
- Model: there exist $B_1, \ldots, B_p$ (fixed, given) such that

$$M = \sum_{j=1}^{p} c_j B_j$$

$\Rightarrow$ find c_j.

Matrix probing questions

- How to recover $\mathbf{c}$?

$$Mz = \sum_{j=1}^{p} c_j B_j z = \Psi_z \, \mathbf{c}$$

  - 1 random realization: Ψ_z has dimension N by p.
  - $q > 1$ random realizations: Ψ_z has dimension Nq by p.

- How large can p get?

- Which B_j?

See Chiu-Demanet, SINUM, 2012 and
Bélanger-Rioux-Demanet preprint (submitted), 2014.

Steps of matrix probing and their complexities

Steps of matrix probing:	**Complexity:**
• Orthonormalize B_j's (QR).	• N^2p^2.
• Build Ψ_z from products $B_j z$.	• N^2pq.
• Obtain Mz.	• q solves of exterior problem.
• Apply pseudoinverse of Ψ_z.	• Np^2q.

Media considered (plots of $c(\mathbf{x})$)

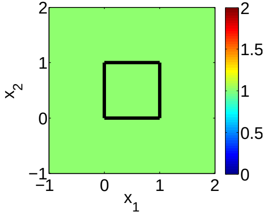


Figure: Uniform.

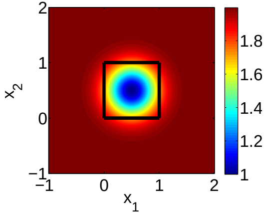


Figure: Slow disk.

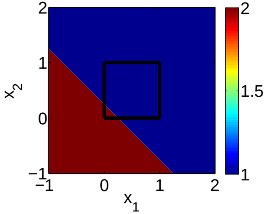


Figure: Diagonal fault.

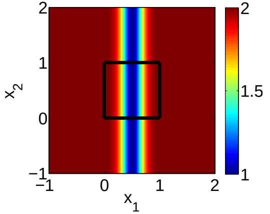


Figure: Waveguide.

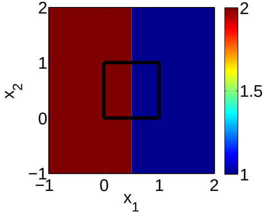


Figure: Vertical fault.

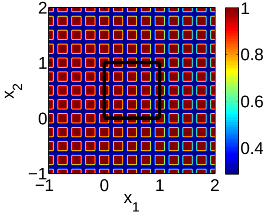


Figure: Periodic.

Real part of solutions u, $\omega = 51.2$, $N = 1024$.

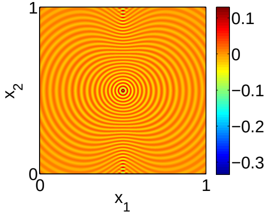


Figure: Waveguide.

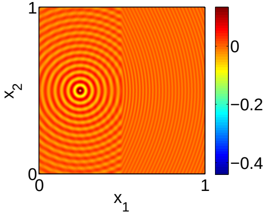


Figure: Vertical, left.

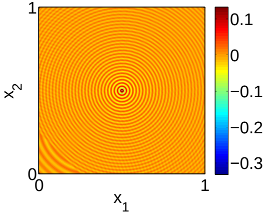


Figure: Diagonal fault.

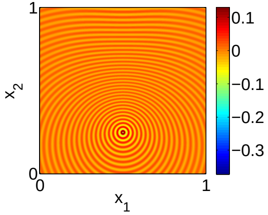


Figure: Slow disk.

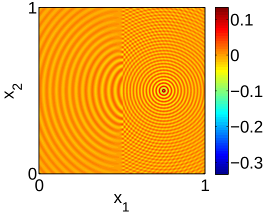


Figure: Vertical, right.

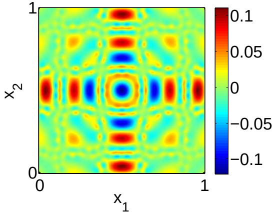


Figure: Periodic.

Probing results

$$D \xrightarrow[\text{expansion}]{\text{probing}} \tilde{D} \xrightarrow[\text{compression}]{\text{PLR}} \overline{D}$$

- Number of basis matrices $p \sim N^{0.2}$ at worst.
- Number of exterior solves q constant as N grows.
- Probing approximation does not degrade with grazing waves.
- Limitations:
  - Easier for smooth media;
  - Careful design of basis matrices needed.

Step 2: Obtain a fast algorithm for matrix-vector products of D

$$D \xrightarrow[\text{expansion}]{\text{probing}} \tilde{D} \xrightarrow[\text{compression}]{\text{PLR}} \overline{D}$$

Intuition: D_{half} numerically low-rank away from singularity

Kernel of the uniform half-space DtN map: $K(r) = \frac{ik^2 H_1^{(1)}(kr)}{2kr}$.

Theorem (RBR, Demanet)

Let $0 < \epsilon \leq 1/2$, and $0 < r_0 < 1$, $r_0 = \Theta(1/k)$. There exists an integer J, functions $\{\Phi_j, \chi_j\}_{j=1}^{J}$ and a number C such that we can approximate $K(|x-y|)$ for $r_0 \leq |x-y| \leq 1$:

$$K(|x-y|) = \sum_{j=1}^{J} \Phi_j(x)\chi_j(y) + E(x,y)$$

where $|E(x,y)| \leq \epsilon$, and $J \leq C\left(\log k \max(|\log \epsilon|, \log k)\right)^2$ with C which does not depend on k or ϵ.

Numerically low-rank ⇒ low-rank matrix block

Function

$$K(|x-y|) = \sum_{j=1}^{J} \Phi_j(x)\chi_j(y),$$

$$K(|x_i-y_\ell|) = \sum_{j=1}^{J} \Phi_j(x_i)\chi_j(y_\ell).$$

Matrix $K_{i\ell} = K(|x_i - y_\ell|)$:

$$K = \sum_{j=1}^{J} \vec{\Phi}_j \ \vec{\chi}_j^{\,*} = \Phi \ \chi^*$$

with $\vec{\Phi}_j$, $\vec{\chi}_j$ the j^{th} columns of matrices Φ, χ.

This is almost the Singular Value Decomposition (SVD) of matrix $K_{i\ell}$.

Proof: D_{half} numerically low-rank away from singularity

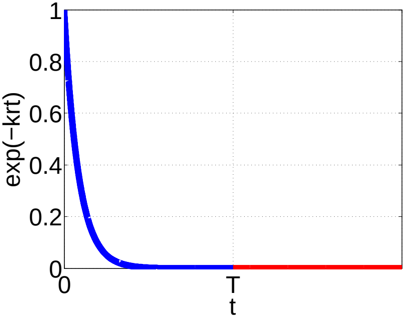
Kernel $K(r) = \frac{ik^2 H_1^{(1)}(kr)}{2kr}$ for uniform half-space DtN map.

Proof: D_{half} numerically low-rank away from singularity

Kernel $K(r) = \frac{ik^2 H_1^{(1)}(kr)}{2kr}$ for uniform half-space DtN map.

$$\frac{1}{kr} = \int_0^\infty e^{-krt} dt \approx \int_0^T e^{-krt} dt \tag{1}$$

with error $\int_T^\infty e^{-krt} dt \leq \epsilon$ for $T = O(|\log \epsilon|)$.



Proof: D_{half} numerically low-rank away from singularity

Use a Gaussian quadrature

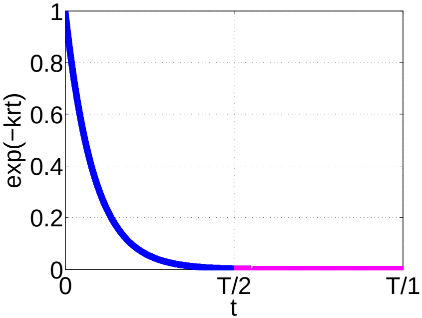
$$\frac{1}{kr} \approx \int_0^T e^{-krt} dt \approx \sum_{j=1}^{n} w_j e^{-krt_j} = \sum_{j=1}^{n} w_j e^{-kxt_j} e^{kyt_j} \qquad x > y$$

Proof: D_{half} numerically low-rank away from singularity

Use a Gaussian quadrature

$$\frac{1}{kr} \approx \int_0^T e^{-krt} dt \approx \sum_{j=1}^{n} w_j e^{-krt_j} = \sum_{j=1}^{n} w_j e^{-kxt_j} e^{kyt_j} \qquad x > y$$

but need a dyadic partition of the interval for convergence.



Proof: D_{half} numerically low-rank away from singularity

- Kernel $K(r) = \frac{ik^2 H_1^{(1)}(kr)}{2kr}$ of uniform half-space DtN map.
- Use Gaussian quadratures for $1/kr$ on dyadic partition of interval:

$$\log k \text{ subintervals, } |\log \epsilon| \text{ pts each}$$

- Treat integral form of Hankel function same way (Martinsson-Rokhlin 2007).
- Multiply $1/kr$ with $H_1^{(1)}$, total number of quad. pts:

$$J \approx (\log k |\log \epsilon|)^2.$$

Partitioned low-rank (PLR) matrices

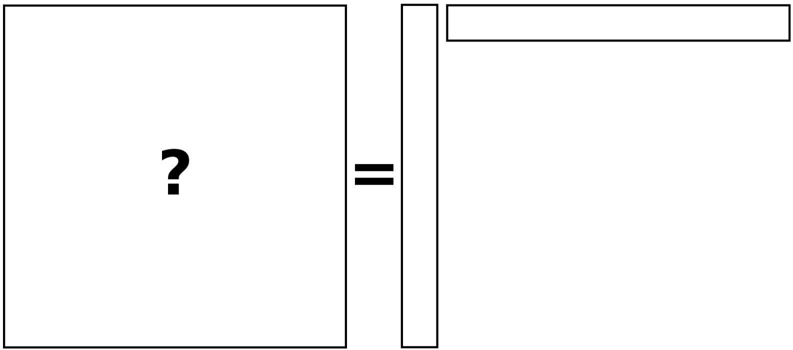
Adaptively, recursively divide blocks of matrix.

Stop when numerical rank $\leq R_{\max}$, with tolerance ϵ.

Partitioned low-rank (PLR) matrices

Adaptively, recursively divide blocks of matrix.

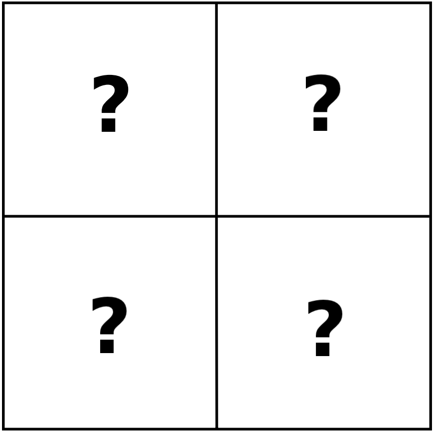
Stop when numerical rank $\leq R_{\max}$, with tolerance ϵ.



Partitioned low-rank (PLR) matrices

Adaptively, recursively divide blocks of matrix.

Stop when numerical rank $\leq R_{\max}$, with tolerance $\epsilon \Rightarrow$ "leaf".



Partitioned low-rank (PLR) matrices

Adaptively, recursively divide blocks of matrix.

Stop when numerical rank $\leq R_{\max}$, with tolerance $\epsilon \Rightarrow$ "leaf".

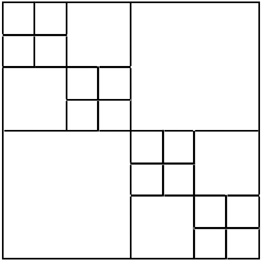


Figure: $\frac{N}{R_{\max}} = 8$, weak hierarchical structure.

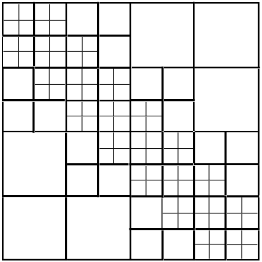


Figure: $\frac{N}{R_{\max}} = 16$, strong hierarchical structure.

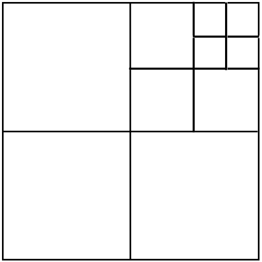


Figure: $\frac{N}{R_{\max}} = 8$, corner PLR structure.

Complexity of compression: PLR matrices

Cost per block B dominated by (randomized) SVD: $O(N_B R_{\max}^2)$.

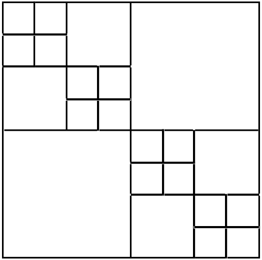


Figure: $\frac{N}{R_{\max}} = 8$, weak h. structure.

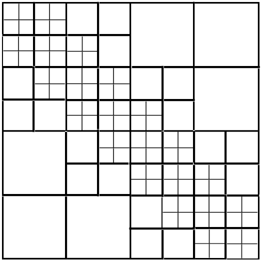


Figure: $\frac{N}{R_{\max}} = 16$, strong h. structure.

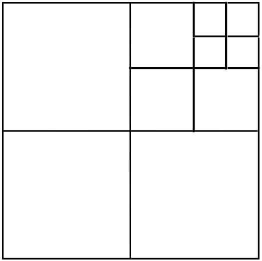


Figure: $\frac{N}{R_{\max}} = 8$, corner PLR structure.

Total complexity:

$O(N R_{\max}^2 \log \frac{N}{R_{\max}})$ $\quad$ $O(N R_{\max}^2 \log \frac{N}{R_{\max}})$ $\quad$ $O(N R_{\max}^2)$

Complexity of matrix-vector products: PLR matrices

Cost per leaf B: $O(N_B R_{\max})$.

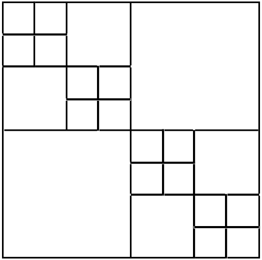


Figure: $\frac{N}{R_{\max}} = 8$, weak h. structure.

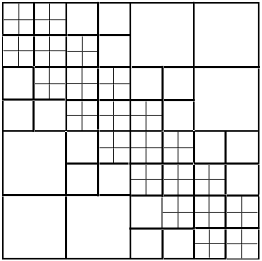


Figure: $\frac{N}{R_{\max}} = 16$, strong h. structure.

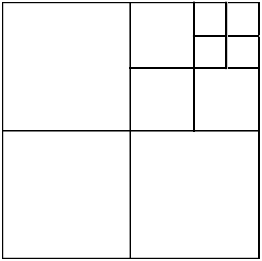


Figure: $\frac{N}{R_{\max}} = 8$, corner PLR structure.

Total complexity:

$O(N R_{\max} \log \frac{N}{R_{\max}})$ $O(N R_{\max} \log \frac{N}{R_{\max}})$ $O(N R_{\max})$

Results of PLR compression after probing

- In general, ask for PLR tolerance

$$\epsilon = \frac{1}{25} \frac{\|D - \tilde{D}\|_F}{\|D\|_F}.$$

Table: $c \equiv 1$

$R_{\max}$	ϵ	$\|D - \overline{D}\|_F / \|D\|_F$	$\|u - \overline{u}\|_F / \|u\|_F$
2	$1.6850e-02$	$4.2126e-01$	$6.5938e-01$
2	$1.6802e-03$	$4.2004e-02$	$7.3655e-02$
2	$5.0068e-05$	$1.2517e-03$	$2.4232e-03$
4	$4.4840e-06$	$1.1210e-04$	$4.0003e-04$
8	$4.3176e-07$	$1.0794e-05$	$1.4305e-05$
8	$2.6198e-08$	$6.5496e-07$	$2.1741e-06$

Results of PLR compression after probing

Table: c is the diagonal fault.

$R_{\max}$	ϵ	$\|D - \overline{D}\|_F / \|D\|_F$	$\|u - \overline{u}\|_F / \|u\|_F$
2	$5.7124e-03$	$1.4281e-01$	$5.3553e-01$
2	$7.6432e-04$	$1.9108e-02$	$7.8969e-02$
4	$1.0241e-04$	$2.5602e-03$	$8.7235e-03$

Table: c is the periodic medium.

$R_{\max}$	ϵ	$\|D - \overline{D}\|_F / \|D\|_F$	$\|u - \overline{u}\|_F / \|u\|_F$
2	$5.1868e-03$	$1.2967e-01$	$2.1162e-01$
2	$1.2242e-03$	$3.0606e-02$	$5.9562e-02$
8	$3.6273e-04$	$9.0682e-03$	$2.6485e-02$

PLR compression after probing

$$D \xrightarrow[\text{expansion}]{\text{probing}} \tilde{D} \xrightarrow[\text{compression}]{\text{PLR}} \overline{D}$$

- Small $R_{\max}$ needed in practice, $R_{\max} \leq 8$.
- Nearly linear matrix-vector product even in heterogeneous media.
- PLR compression is very flexible, "one size fits all".

Conclusion – so far

- Insights from half-space DtN map to expand then compress exterior DtN map
- Handful of PDE solves $\Rightarrow$ exterior DtN map to good accuracy $\Rightarrow$ HE solution to good accuracy
- Compressed DtN map $\Rightarrow$ fast matrix-vector products

Conclusion – complexities

Constructing D:

- Matrix probing expansion, assuming fast solver.
- PLR compression.

Complexity:

- $\sim q(N+w)^2, \qquad q \leq 50.$
- $\sim NR_{\max}^2, \qquad R_{\max} \leq 8.$

Applying D:

- Dense matrix-vector product.
- PLR matrix-vector product.

Complexity:

- $\sim 16N^2.$
- $\sim 4NR_{\max} \log \frac{N}{R_{\max}} + 12NR_{\max}.$

Conclusion – outlook

- 3D
- Probe (and compress) entire structure of the Green's function?
- Integrate in Domain Decomposition Methods